

Lasserre SDPs, ℓ_1 -embeddings, and approximating non-uniform sparsest cut via generalized spectra

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Abstract

We give an approximation algorithm for non-uniform sparsest cut with the following guarantee: For any $\varepsilon, \delta \in (0, 1)$, given cost and demand graphs with edge weights $C, D : \binom{V}{2} \rightarrow \mathbb{R}_+$ respectively, we can find a set $T \subseteq V$ with $\frac{C(T, V \setminus T)}{D(T, V \setminus T)}$ at most $\frac{1+\varepsilon}{\delta}$ times the optimal non-uniform sparsest cut value, in time $2^{r/(\delta\varepsilon)} \text{poly}(n)$ provided $\lambda_r \geq \Phi^*/(1-\delta)$. Here λ_r is the r 'th smallest generalized eigenvalue of the Laplacian matrices of cost and demand graphs; $C(T, V \setminus T)$ (resp. $D(T, V \setminus T)$) is the weight of edges crossing the $(T, V \setminus T)$ cut in cost (resp. demand) graph and Φ^* is the sparsity of the optimal cut. In words, we show that the non-uniform sparsest cut problem is easy when the generalized spectrum grows moderately fast. To the best of our knowledge, there were no results based on higher order spectra for non-uniform sparsest cut prior to this work.

Even for uniform sparsest cut, the quantitative aspects of our result are somewhat stronger than previous methods. Similar results hold for other expansion measures like edge expansion, normalized cut, and conductance, with the r 'th smallest eigenvalue of the *normalized* Laplacian playing the role of $\lambda_r(G)$ in the latter two cases.

Our proof is based on an ℓ_1 -embedding of vectors from a semi-definite program from the Lasserre hierarchy. The embedded vectors are then rounded to a cut using standard threshold rounding. We hope that the ideas connecting ℓ_1 -embeddings to Lasserre SDPs will find other applications. Another aspect of the analysis is the adaptation of the column selection paradigm from our earlier work on rounding Lasserre SDPs [GS11] to pick a set of *edges* rather than vertices. This feature is important in order to extend the algorithms to non-uniform sparsest cut.

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1 Introduction

The problem of finding sparsest cut on graphs is a fundamental optimization problem that has been intensively studied. The problem is inherently interesting, and is important as a building block for divide-and-conquer algorithms on graphs as well as to many applications such as image segmentation [SM00, SG07], VLSI layout [BL84], packet routing in distributed networks [AP90], etc.

Let us define the prototypical sparsest cut problem more concretely. We are given a set of n -vertices, V , along with two functions $C, D : \binom{V}{2} \rightarrow \mathbb{R}_+$ representing edge weights of some cost and demand graphs, respectively. Then given any subset $T \subset V$, we define its *sparsity* as the following ratio:

$$\Phi_T \stackrel{\text{def}}{=} \frac{\sum_{u < v} C_{u,v} \cdot |\mathbb{1}_T(u) - \mathbb{1}_T(v)|}{\sum_{u < v} D_{u,v} \cdot |\mathbb{1}_T(u) - \mathbb{1}_T(v)|}, \quad (1)$$

where $\mathbb{1}_T$ is the indicator function of T . Our goal in the SPARSEST CUT problem is to find a subset $T \subset V$ with minimum sparsity, which we denote by $\Phi^* \stackrel{\text{def}}{=} \min_{T \subset V} \Phi_T$. The special case of demand graph being a clique, where the denominator of eq. (1) becomes $|T| \cdot |V \setminus T|$, is called the UNIFORM SPARSEST CUT problem.

The value of the sparsest cut can be understood in terms of the spectral properties of cost and demand graphs. Let $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$ be the *generalized eigenvalues* between the Laplacian matrices of cost and demand graphs (see Section 2 for formal definitions). In a way similar to the “easy” direction of Cheeger’s inequality, we can use Courant-Fischer Theorem to show that $\lambda_1 \leq \Phi^*$. At some point, the eigenvalue λ_r will exceed Φ^* . Our main result is an approximation algorithm for SPARSEST CUT which is efficient when this happens for small r . In particular:

Corollary 1 (See Corollary 23). *Given V and $C, D : \binom{V}{2} \rightarrow \mathbb{R}_+$, for any positive integer r , one of the following holds.*

- Either one can find $T \subset V$ with $\Phi_T \leq 2\Phi^*$ in time $2^{O(r)} \text{poly}(n)$ where $n = |V|$,
- Or $\Phi^* \geq 0.49\lambda_r$.

Our actual approximation guarantee is stronger and is stated in Corollary 23 (the above follows as a corollary with suitable choice of parameters). We can also get similar results for various expansion problems such as normalized cut, edge expansion and conductance using the same algorithm.

1.1 Previous approximation algorithms for sparsest cut

As the (UNIFORM) SPARSEST CUT problem and closely related variants (such as edge expansion and conductance) are all NP-hard in general, theoretically much effort has gone into the design of good approximation algorithm for the problem.

For UNIFORM SPARSEST CUT problem, the hard direction of Cheeger’s inequality shows one can “round” the eigenvector corresponding to λ_1 to a cut T satisfying $\Phi_T \leq \sqrt{8d_{\max}\lambda_1(G)}$ where d_{\max} is the maximum degree. This gives $O(\sqrt{d_{\max}/\Phi^*(G)}) \leq O(\sqrt{d_{\max}/\lambda_1(G)})$ approximation which is good for moderate values of Φ^* for the case of UNIFORM SPARSEST CUT. To the best of our knowledge, no analogue of this result is known for SPARSEST CUT.

For smaller values of Φ^* , the best approximation for SPARSEST CUT is based on solving a convex relaxation of the problem, and then rounding the solution to a cut. Using linear programming (LP), in a seminal work, Leighton and Rao [LR88] gave a factor $O(\log n)$ approximation for SPARSEST CUT (here n denotes the number of vertices). Beautiful connections of approximating sparsest cut to embeddings of metric spaces into the ℓ_1 -metric were later discovered in [LLR95, AR98].

Using a semi-definite programming (SDP) relaxation, the approximation ratio was improved to $O(\sqrt{\log n})$ for UNIFORM SPARSEST CUT in the breakthrough work [ARV09]. For SPARSEST CUT, using ℓ_1 embeddings of negative type metrics, an approximation factor of $O(\log^{3/4} n)$ was obtained in [CGR08] and a factor $O(\sqrt{\log n \log \log n})$, nearly matching the UNIFORM SPARSEST CUT case, was obtained in [ALN08].

Recently, higher order eigenvalues were used to approximate many graph partitioning problems. In [GS11], we gave an algorithm based on SDPs from the Lasserre hierarchy achieving an approximation factor of the form $(1 + \varepsilon) / \min\{1, \tilde{\lambda}_r\}$ for problems such as minimum bisection, small set expansion, etc, where $\tilde{\lambda}_r$ is the r 'th smallest eigenvalue of the normalized Laplacian. On a similar front, for the UNIFORM SPARSEST CUT problem, if the r^{th} eigenvalue is large relative to expansion, we show how to combine the eigenspace enumeration of [ABS10] with a cut improvement procedure from [AL08]¹ to obtain a constant factor approximation for UNIFORM SPARSEST CUT in time $n^{O(1)}2^{O(r)}$. The details of this combination are briefly spelled out in Appendix B. We will revisit this approach in Section 1.2 to show why it does not work for SPARSEST CUT.

A common theme in this line of work is that one can obtain a constant factor approximation with running time being a function of how fast the spectrum grows (both our algorithms in this paper and the ones in [GS11] in fact allow approximation schemes). Put differently, one can identify a generic condition which highlights what kind of graphs are easy.

To the best of our knowledge, in the case of SPARSEST CUT with an arbitrary demand graph, no such results of the above vein are known. In fact, we are not aware of the analog of the harder direction of Cheeger's inequality, let alone spectrum based approximation schemes. In this paper, we present such an approximation scheme based on the generalized eigenvalues.

1.2 Overview of Our Contributions

In this section, we briefly describe our main contributions in terms algorithmic tools and techniques over similar algorithms such as [GS11].

Main Contributions. Our algorithm is based on solving one of the strongest known SDP relaxations, r -rounds of Lasserre Hierarchy, similar to [GS11]. Any solution for this SDP yields a vector for each r -subset of vertices and each possible labeling of them. The rounding algorithm in [GS11] is based on choosing a set of r -nodes, "seeds", then labeling these using the SDP solution. Finally these labels are "propagated" to other vertices **independently at random**. Such rounding is acceptable for constraint satisfaction type problems such as maximum cut.

Unfortunately for problems such as SPARSEST CUT, independent rounding is too "crude": It tends to break the graph into many disconnected components, which is rather disastrous for SPARSEST CUT.

In this paper, we consider a more "delicate" rounding based on thresholding. Our main contribution is to show how the performance of such rounding is related to some strong geometrical quantities of underlying SDP solution, and we show how to bound it using generalized spectra.

Comparison with Subspace Enumeration. One successful technique for designing approximation algorithms based on higher order spectrum is subspace enumeration [Kol10, ABS10]. Suppose we have a target set T corresponding to a UNIFORM SPARSEST CUT. These techniques rely on the fact that the indicator vector T should have a large component on the span of small eigenvectors. Thus by enumerating over the vectors on this subspace using some ε -net, we can find a set whose symmetric difference with T is small. Combining this with a cut improvement algorithm due to [AL08], we can obtain an approximation algorithm for UNIFORM SPARSEST CUT problem with slightly worse approximation factors than ours (see Appendix B).

¹We thank anonymous reviewers for bringing this paper to our attention.

Unfortunately the immediate extension of this approach to SPARSEST CUT by using the generalized eigenvectors does not work as the generalized eigenvectors are not *orthogonal* in the Euclidean space.

2 Preliminaries

Sets. Let $[m] \stackrel{\text{def}}{=} \{1, 2, \dots, m\}$. Given set A and positive integer k , we use $\binom{A}{k}$ (resp. $\binom{A}{\leq k}$) to denote the set of all possible size k (resp. size at most k) subsets of A . We use \mathbb{R}_+ to denote the set of non-negative reals.

Euclidean Space. Given $A \subseteq \mathbb{R}$ and row set B , we use A^B to denote the set of vectors where each row (axis) is associated with an element of B , and each coordinate is chosen from A . For any vector $x \in A^B$, its coordinate at axis $b \in B$ is denoted by $x(b)$. Let $\|x\|_p$ be its p^{th} norm with $\|x\| \stackrel{\text{def}}{=} \|x\|_2$, and x^T be its transpose. Finally for any $x, y \in \mathbb{R}^B$, let $\langle x, y \rangle = x^T y$ be their inner product $\sum_{b \in B} x_b y_b$.

Matrices. Given $A \subseteq \mathbb{R}$, row set B and column set C , we use $A^{B,C}$ to denote the set of matrices with rows and columns associated with elements of B and C , respectively. Given matrix $X \in A^{B,C}$, for any $b \in B, c \in C$, we will use $X_{b,c} \in A$ to denote entry of X at row b and column c . For convenience, we use $X_c \in A^B$ to denote the vector corresponding to the column c of X . Likewise given subset of columns of X , $S \subseteq C$, we use $X_S \in A^{B,S}$ to denote the matrix corresponding to the columns S of X . Given matrix X , we use $\|X\|_F$, $\text{Tr}(X)$ and X^T to denote Frobenius norm of X , its trace and transpose.

Finally we use X^Π and X^\perp to denote the projection matrices onto the span of X and its orthogonal complement.

Positive Semi-Definite (PSD) Ordering. Given a symmetric matrix $X \in \mathbb{R}^{A,A}$, we say X is a PSD matrix, denoted by $X \succeq 0$, iff $y^T X y \geq 0$ for all $y \in \mathbb{R}^A$.

Eigenvalues. Given symmetric matrix $X \in \mathbb{R}^{A,A}$, for any integer $i \leq |A|$, we define its i^{th} smallest and largest eigenvalues as the following, respectively:

$$\lambda_i \stackrel{\text{def}}{=} \max_{\text{rank}(Z) \leq i-1} \min_{w \perp Z, w \neq 0} \frac{w^T X w}{w^T w}, \quad \sigma_i \stackrel{\text{def}}{=} \min_{\text{rank}(Z) \leq i-1} \max_{w \perp Z, w \neq 0} \frac{w^T X w}{w^T w}.$$

Generalized Eigenvalues. Given two symmetric matrices $X, Y \in \mathbb{R}^{A,A}$ with $Y \succeq 0$, for any integer $i \leq \text{rank}(Y)$, we define their i^{th} smallest generalized eigenvalue as the following:

$$\lambda_i \stackrel{\text{def}}{=} \max_{\text{rank}(Z) \leq i-1} \min_{w \perp Z, Y w \neq 0} \frac{w^T X w}{w^T Y w}.$$

Graphs. We assume all graphs are simple, undirected and edge-weighted with non-negative weights. We associate each graph with its edge weight function of the form $W : \binom{V}{2} \rightarrow \mathbb{R}_+$, where we use $W_{u,v}$ to denote the weight of edge between u and v for convenience.

Laplacian Matrices. Given a graph with weights $W : \binom{V}{2} \rightarrow \mathbb{R}_+$, the associated graph Laplacian matrix, $L_W \in \mathbb{R}^{V,V}$, is defined as the following symmetric matrix:

$$(L_W)_{a,b} = \begin{cases} \sum_c W_{a,c} & \text{if } a = b, \\ -W_{a,b} & \text{if } a \neq b. \end{cases}$$

The following identity is trivial, from which we can easily see that $L_W \succeq 0$:

$$X \in \mathbb{R}^{V,V} \implies \text{Tr} [X^T X L_W] = \sum_{u < v} W_{u,v} \|X_u - X_v\|^2.$$

2.1 Lasserre Hierarchy

We present the formal definitions of the Lasserre Hierarchy of SDP relaxations [Las02], tailored to the setting of the problems we are interested in, where the goal is to assign to each vertex/variable from V a label from $\{0, 1\}$.

Definition 2 (Lasserre vector set). *Given a set of variables V and a positive integer r , a collection of vectors x is said to satisfy r -rounds of Lasserre Hierarchy, denoted by $x \in \text{Lasserre}_r(V)$, if it satisfies the following conditions:*

1. For each set $S \in \binom{V}{\leq r+1}$, there exists a function $x_S : \{0, 1\}^S \rightarrow \mathbb{R}^\Upsilon$ that associates a vector of some finite dimension Υ with each possible labeling of S . We use $x_S(f)$ to denote the vector associated with the labeling $f \in [k]^S$. For singletons $u \in V$, we will use x_u and $x_u(1)$ interchangeably.
For $f \in \{0, 1\}^S$ and $v \in S$, we use $f(v)$ as the label v receives from f . Also given sets S with labeling $f \in \{0, 1\}^S$ and T with labeling $g \in \{0, 1\}^T$ such that f and g agree on $S \cap T$, we use $f \circ g$ to denote the labeling of $S \cup T$ consistent with f and g : If $u \in S$, $(f \circ g)(u) = f(u)$ and vice versa.
2. $x_\emptyset \neq 0$.
3. $\langle x_S(f), x_T(g) \rangle = 0$ if there exists $u \in S \cap T$ such that $f(u) \neq g(u)$.
4. $\langle x_S(f), x_T(g) \rangle = \langle x_A(f'), x_B(g') \rangle$ if $S \cup T = A \cup B$ and $f \circ g = f' \circ g'$.
5. For any $u \in V$, $\sum_{j \in \{0, 1\}} \|x_u(j)\|^2 = \|x_\emptyset\|^2$.
6. (implied by above constraints) For any $S \in \binom{V}{\leq r+1}$, $u \in S$ and $f \in \{0, 1\}^{S \setminus \{u\}}$, $\sum_{g \in \{0, 1\}^u} x_S(f \circ g) = x_{S \setminus \{u\}}(f)$.

3 Our Algorithm and Its Analysis

The complete algorithm is presented in Algorithm 3. It is based on rounding a certain r' -rounds of Lasserre Hierarchy relaxation for the SPARSEST CUT problem given positive integer r' :

$$\min \frac{\sum_{u < v} C_{u,v} \|x_u - x_v\|^2}{\sum_{u < v} D_{u,v} \|x_u - x_v\|^2} \quad \text{st} \quad \sum_{u < v} D_{u,v} \|x_u - x_v\|^2 > 0, \quad x \in \text{Lasserre}_{r'}(V), \quad \|x_\emptyset\|^2 = 1. \quad (2)$$

It is easy to see that eq. (2) is indeed a relaxation of SPARSEST CUT problem. Even though it is not an SDP problem (it is quasi-convex), there is an equivalent SDP formulation.

Lemma 3. *The following SDP is equivalent to eq. (2):*

$$\min \sum_{u < v} C_{u,v} \|w_u - w_v\|^2 \quad \text{st} \quad \sum_{u < v} D_{u,v} \|w_u - w_v\|^2 = 1, \quad \|w_\emptyset\|^2 > 0, \quad w \in \text{Lasserre}_{r'}(V). \quad (3)$$

Remark 4. *The constraint $\|w_\emptyset\|^2 > 0$ in eq. (3) is redundant, but we included it for the sake of clarity.*

Proof of Lemma 3. Given a feasible solution x of eq. (2), consider the following collection of vectors, $w = [w_T]_{T \in \binom{V}{\leq r'}}$. For each $T \in \binom{V}{\leq r'}$, we define w_T as $w_T \stackrel{\text{def}}{=} \frac{1}{\sqrt{\sum_{u < v} D_{u,v} \|x_u - x_v\|^2}} x_T$. It is easy to see that $\sum_{u < v} D_{u,v} \|w_u - w_v\|^2 = 1$ and objective values are equal. Finally $\|w_\emptyset\|^2 = \frac{1}{\sum_{u < v} D_{u,v} \|x_u - x_v\|^2} > 0$ since $0 < \sum_{u < v} D_{u,v} \|x_u - x_v\|^2 < +\infty$.

For the other direction of equivalence, suppose w is a feasible solution of eq. (3). For each $T \in \binom{V}{\leq r'}$, $f \in \{0, 1\}^T$, let $x_T(f) \leftarrow \frac{1}{\|w_\emptyset\|} w_T(f)$. It is easy to see that the objective values are equal. Rest of the proof for x being a feasible solution of eq. (2) follows in the same way with the previous direction. \square

Remark 5. Main components of our rounding, Algorithms 1 and 2, are scale invariant; thus the formulation given in eq. (3) is sufficient for rounding purposes. But we chose to first present eq. (2) as it is more intuitive.

3.1 Intuition Behind Our Rounding

For an intuition behind our rounding procedure, presented in Algorithm 1, we start with a simple rounding procedure, which is based on the seed based propagation framework from [GS11]. Later in this section, we will show how to fix it.

First Attempt. Consider the following procedure. On input $x \in \text{Lasserre}_{2r'+2}(V)$:

1. Choose a set of r' -edges from the demand graph, say $\mathcal{S} \subseteq \binom{V}{2}$ (seed edges).
2. Let $\tilde{\mathcal{S}}$ be the set of their endpoints, $\tilde{\mathcal{S}} \leftarrow \{u \in V \mid \text{exists } v \text{ such that } \{u, v\} \in \mathcal{S}\} \subseteq V$.
3. Observe that $|\tilde{\mathcal{S}}| \leq 2r'$, hence the values $\|x_{\tilde{\mathcal{S}}}(f)\|^2$ define a probability distribution over all labelings of $\tilde{\mathcal{S}}$, $f : \tilde{\mathcal{S}} \rightarrow \{0, 1\}$. So sample a labeling for $\tilde{\mathcal{S}}$, $f : \tilde{\mathcal{S}} \rightarrow \{0, 1\}$, with probability $\|x_{\tilde{\mathcal{S}}}(f)\|^2$.
4. Choose a threshold $\tau \in [0, 1]$ uniformly at random and output the following set:

$$T(f, \tau) \stackrel{\text{def}}{=} \left\{ u \in V \mid \frac{\langle x_{\tilde{\mathcal{S}}}(f), x_u \rangle}{\|x_{\tilde{\mathcal{S}}}(f)\|^2} \geq \tau \right\}.$$

In order for this procedure to make sense, the range of $\frac{\langle x_{\tilde{\mathcal{S}}}(f), x_u \rangle}{\|x_{\tilde{\mathcal{S}}}(f)\|^2}$ should be similar to τ 's range. In the following claim, we prove this.

Claim 6. *Provided that $x_{\tilde{\mathcal{S}}}(f) \neq 0$, we have: (i) $0 \leq \frac{\langle x_{\tilde{\mathcal{S}}}(f), x_u \rangle}{\|x_{\tilde{\mathcal{S}}}(f)\|^2} \leq 1$ for any $u \in V$, (ii) $\frac{|\langle x_{\tilde{\mathcal{S}}}(f), x_u - x_v \rangle|}{\|x_{\tilde{\mathcal{S}}}(f)\|^2} \leq 1$ for any pair $u, v \in V$, (iii) $\frac{\langle x_{\tilde{\mathcal{S}}}(f), x_u \rangle}{\|x_{\tilde{\mathcal{S}}}(f)\|^2} = f(u)$ for any $u \in \tilde{\mathcal{S}}$.*

Proof of (i) and (ii). We will only prove (i), from which (ii) follows immediately. The lower bound follows from $\langle x_{\tilde{\mathcal{S}}}(f), x_u \rangle = \|x_{\tilde{\mathcal{S}} \cup \{u\}}(f \circ 1)\|^2 \geq 0$. For the upper bound, we have:

$$\begin{aligned} \langle x_{\tilde{\mathcal{S}}}(f), x_{\tilde{\mathcal{S}}}(f) - x_u \rangle &= \|x_{\tilde{\mathcal{S}}}(f)\|^2 - \langle x_{\tilde{\mathcal{S}}}(f), x_u \rangle = \langle x_{\tilde{\mathcal{S}}}(f), x_\emptyset \rangle - \langle x_{\tilde{\mathcal{S}}}(f), x_u \rangle \\ &= \langle x_{\tilde{\mathcal{S}}}(f), x_u(0) \rangle = \|x_{\tilde{\mathcal{S}} \cup \{u\}}(f \circ 0)\|^2 \geq 0. \end{aligned} \quad \square$$

Proof of (iii). Follows from the fact that $\langle x_{\tilde{\mathcal{S}}}(f), x_u(f(u)) \rangle = \|x_{\tilde{\mathcal{S}}}(f)\|^2$. \square

Let's calculate the probability of separating two vertices by this procedure.

Claim 7. $\mathbb{E}_{f, \tau} \left[\left| \mathbb{1}_{T(f, \tau)}(u) - \mathbb{1}_{T(f, \tau)}(v) \right| \right] = \sum_f |\langle x_{\tilde{\mathcal{S}}}(f), x_u - x_v \rangle|.$

Proof. For fixed f , by Claim 6 the probability of separating u and v is equal to $\frac{|\langle x_{\tilde{\mathcal{S}}}(f), x_u - x_v \rangle|}{\|x_{\tilde{\mathcal{S}}}(f)\|^2}$. Taking expectation over f :

$$\mathbb{E}_{f, \tau} \left[\left| \mathbb{1}_{T(f, \tau)}(u) - \mathbb{1}_{T(f, \tau)}(v) \right| \right] = \sum_f \|x_{\tilde{\mathcal{S}}}(f)\|^2 \frac{|\langle x_{\tilde{\mathcal{S}}}(f), x_u - x_v \rangle|}{\|x_{\tilde{\mathcal{S}}}(f)\|^2} = \sum_f |\langle x_{\tilde{\mathcal{S}}}(f), x_u - x_v \rangle|. \quad \square$$

Second Attempt. For any fixed $f : \tilde{\mathcal{S}} \rightarrow \{0, 1\}$, there are at most n different $T(f, \tau)$'s. Hence instead of choosing $f : \tilde{\mathcal{S}} \rightarrow \{0, 1\}$ and $\tau \in [0, 1]$ randomly, we can perform an exhaustive search over all possible such sets and output the one with minimum sparsity. Since there are at most $n2^{O(r')}$ many unique $T(f, \tau)$'s, the exhaustive search can easily be implemented in time $\text{poly}(n)2^{O(r')}$. The rounding procedure along with this modification is presented in Algorithm 1.

3.2 Seed Based ℓ_1 -embedding

We choose our embedding so as to reflect the rounding procedure outlined in the previous section.

Definition 8 (Seed Based Embedding). *Given $x \in \text{Lasserre}_{2r'+2}(V)$ and $\mathcal{S} \subseteq \binom{V}{2}$ with $|\mathcal{S}| \leq r'$, let $\tilde{\mathcal{S}}$ be the endpoints of edges in \mathcal{S} so that $\mathcal{S} \subseteq \binom{\tilde{\mathcal{S}}}{2}$. Then we define the seed based embedding of x as the following collection of vectors. For each $u \in V$, $y_u^{\mathcal{S}} \in \mathbb{R}^{\{0,1\}^{\tilde{\mathcal{S}}}}$ is given by $y_u^{\mathcal{S}} \stackrel{\text{def}}{=} \left[\langle x_{\tilde{\mathcal{S}}}(f), x_u \rangle \right]_{f: \tilde{\mathcal{S}} \rightarrow \{0,1\}}$.*

Observe that $\|y_u^{\mathcal{S}} - y_v^{\mathcal{S}}\|_1$ is equal to the probability that u and v are separated as shown in Claim 7.

It is well known that once we have an ℓ_1 -embedding, we can get a cut with similar sparsity by choosing the best threshold cut along each coordinate and this is exactly what we do in Algorithm 1. See Appendix A for a proof of the following well-known lemma.

Lemma 9 (ℓ_1 Embeddings and Threshold Cuts [LLR95]). *Given a set of vertices V , a collection of vectors $\left[y_u \in \mathbb{R}^{\Upsilon} \right]_{u \in V}$ representing an embedding of V , the following holds. For any $C, D : \binom{V}{2} \rightarrow \mathbb{R}_+$ being the edge weights of graphs G and H , respectively:*

$$\min_{\substack{f \in \Upsilon, \\ \tau \in \mathbb{R}}} \Phi_{T(f, \tau)} \leq \frac{\sum_{u < v} C_{u,v} \|y_u - y_v\|_1}{\sum_{u < v} D_{u,v} \|y_u - y_v\|_1}. \quad (4)$$

Here $T(f, \tau) \stackrel{\text{def}}{=} \{u \in V \mid y_u(i) \geq \tau\}$ represents the threshold cut along coordinate $f \in \Upsilon$.

In the rest of this section, we will upper bound eq. (4) for our embedding from Definition 8.

Claim 10. $\|y_u^{\mathcal{S}} - y_v^{\mathcal{S}}\|_1 \leq \|x_u - x_v\|^2$.

Proof. Since $x \in \text{Lasserre}_{2r'+2}(V)$, we can express x_u and x_v as:

$$x_u = x_{u,v}(10) + x_{u,v}(11), \quad x_v = x_{u,v}(01) + x_{u,v}(11) \implies x_u - x_v = x_{u,v}(10) - x_{u,v}(01).$$

The following identity follows easily²:

$$\|x_u - x_v\|^2 = \|x_{u,v}(10)\|^2 + \|x_{u,v}(01)\|^2 - 2 \underbrace{\langle x_{u,v}(10), x_{u,v}(01) \rangle}_{=0} = \|x_{u,v}(10)\|^2 + \|x_{u,v}(01)\|^2. \quad (5)$$

Therefore:

$$\begin{aligned} |y_u^{\mathcal{S}}(f) - y_v^{\mathcal{S}}(f)| &= |\langle x_{\tilde{\mathcal{S}}}(f), x_u - x_v \rangle| = |\langle x_{\tilde{\mathcal{S}}}(f), x_{u,v}(10) - x_{u,v}(01) \rangle| \\ &\leq |\langle x_{\tilde{\mathcal{S}}}(f), x_{u,v}(10) \rangle| + |\langle x_{\tilde{\mathcal{S}}}(f), x_{u,v}(01) \rangle| \end{aligned}$$

² Intuitively, it corresponds to the following. The “probability” of u and v are separated is equal to the probability of u and v being labeled with 1 and 0 or 0 and 1.

For any $g : \{u, v\} \rightarrow \{0, 1\}$, $\langle x_{\tilde{S}}(f), x_{u,v}(g) \rangle = \|x_{\tilde{S} \cup \{u,v\}}(f \circ g)\|^2 \geq 0$. Thus:

$$= \langle x_{\tilde{S}}(f), x_{u,v}(10) \rangle + \langle x_{\tilde{S}}(f), x_{u,v}(01) \rangle.$$

Summing over f and using the fact that $x_\emptyset = \sum_f x_{\tilde{S}}(f)$:

$$\begin{aligned} \|y_u^S - y_v^S\|_1 &\leq \left\langle \sum_f x_{\tilde{S}}(f), x_{u,v}(10) + x_{u,v}(01) \right\rangle = \langle x_\emptyset, x_{u,v}(10) + x_{u,v}(01) \rangle \\ &= \|x_{u,v}(10)\|^2 + \|x_{u,v}(01)\|^2 = \|x_u - x_v\|^2 \text{ by eq. (5).} \end{aligned} \quad \square$$

Claim 11. $\|y_u^S - y_v^S\|_1 \geq \sum_{f: x_{\tilde{S}}(f) \neq 0} \langle \overline{x_{\tilde{S}}(f)}, x_u - x_v \rangle^2$ where $\overline{x_{\tilde{S}}(f)} \stackrel{\text{def}}{=} \frac{x_{\tilde{S}}(f)}{\|x_{\tilde{S}}(f)\|}$ is the unit vector for $x_{\tilde{S}}(f)$.

Proof. For any $f : x_{\tilde{S}}(f) \neq 0$, by Claim 6, $0 \leq \frac{|\langle x_{\tilde{S}}(f), x_u - x_v \rangle|}{\|x_{\tilde{S}}(f)\|^2} \leq 1$ thus $\frac{|\langle x_{\tilde{S}}(f), x_u - x_v \rangle|}{\|x_{\tilde{S}}(f)\|^2} \geq \left(\frac{\langle x_{\tilde{S}}(f), x_u - x_v \rangle}{\|x_{\tilde{S}}(f)\|^2} \right)^2$.

Multiplying both sides with $\|x_{\tilde{S}}(f)\|^2 > 0$, we obtain $|\langle x_{\tilde{S}}(f), x_u - x_v \rangle| \geq \frac{\langle x_{\tilde{S}}(f), x_u - x_v \rangle^2}{\|x_{\tilde{S}}(f)\|^2}$.

Summing over all $f : x_{\tilde{S}}(f) \neq 0$, we obtain the desired lower bound, $\|y_u^S - y_v^S\|_1 = \sum_f |\langle x_{\tilde{S}}(f), x_u - x_v \rangle| \geq \sum_{f: x_{\tilde{S}}(f) \neq 0} \frac{\langle x_{\tilde{S}}(f), x_u - x_v \rangle^2}{\|x_{\tilde{S}}(f)\|^2}$. \square

In its current form, our lower bound is not very useful as it involves the *higher order* vectors ($x_{\tilde{S}}(f)$'s) from our relaxation. Unfortunately these vectors are very hard to reason about: We do not have any direct handle on them. Therefore our goal is to relate this expression to some other expression that only involves the vectors for edges ($x_u - x_v$'s). We first introduce some notation.

Notation 12. Let $\Pi_{\tilde{S}} \stackrel{\text{def}}{=} \sum_{f: x_{\tilde{S}}(f) \neq 0} \overline{x_{\tilde{S}}(f)} \cdot \overline{x_{\tilde{S}}(f)}^T$

We can rewrite the lower bound from Claim 11 in terms of $\Pi_{\tilde{S}}$ as follows:

$$\sum_f \langle \overline{x_{\tilde{S}}(f)}, x_u - x_v \rangle^2 = (x_u - x_v)^T \left(\sum_f \overline{x_{\tilde{S}}(f)} \cdot \overline{x_{\tilde{S}}(f)}^T \right) (x_u - x_v) = (x_u - x_v)^T \Pi_{\tilde{S}} (x_u - x_v). \quad (6)$$

It was first observed in [GS11] that $\Pi_{\tilde{S}}$ has a special structure – it is a projection matrix onto the span of vectors $\{x_{\tilde{S}}(f)\}_f$.

Proposition 13. $\Pi_{\tilde{S}}^2 = \Pi_{\tilde{S}}$, i.e. $\Pi_{\tilde{S}}$ is a projection matrix onto the span of vectors in $\{x_{\tilde{S}}(f)\}$.

Proof. Observe that $\langle \overline{x_{\tilde{S}}(f)}, \overline{x_{\tilde{S}}(g)} \rangle = \begin{cases} 1 & \text{if } f = g, \\ 0 & \text{else} \end{cases}$. Then we have:

$$\Pi_{\tilde{S}}^2 = \sum_{f,g} \langle \overline{x_{\tilde{S}}(f)}, \overline{x_{\tilde{S}}(g)} \rangle \overline{x_{\tilde{S}}(f)} \cdot \overline{x_{\tilde{S}}(g)}^T = \sum_f \overline{x_{\tilde{S}}(f)} \cdot \overline{x_{\tilde{S}}(f)}^T = \Pi_{\tilde{S}}. \quad \square$$

For each seed edge $\{u, v\} \in \mathcal{S}$, $x_u - x_v \in \text{span}\{x_{\tilde{S}}(f)\}$. This means we can lower bound the matrix $\Pi_{\tilde{S}}$ in terms of the projection matrix onto the span of vectors corresponding to seed edges!

Notation 14. Let $P_{\mathcal{S}}$ be the projection matrix onto the span of $\{x_u - x_v\}_{\{u,v\} \in \mathcal{S}}$. Similarly let $P_{\mathcal{S}}^\perp$ be projection matrix onto the orthogonal complement of $\{x_u - x_v\}_{\{u,v\} \in \mathcal{S}}$, i.e., $P_{\mathcal{S}}^\perp = I - P_{\mathcal{S}}$. Here I is the identity matrix.

Lemma 15. $\|y_u^S - y_v^S\|_1 \geq \|P_S(x_u - x_v)\|^2 = \|x_u - x_v\|^2 - \|P_S^\perp(x_u - x_v)\|^2$.

Proof. From Claim 11 and eq. (6) we see that $\|y_u^S - y_v^S\|_1 \geq (x_u - x_v)^T \Pi_{\tilde{S}}(x_u - x_v)$. For any $u \in \tilde{S}$, $x_u = \sum_{f: f(u)=1} x_{\tilde{S}}(f)$ hence $x_u \in \text{span}\{x_{\tilde{S}}(f)\}$. In particular, for any pair $u, v \in \tilde{S}$: $x_u - x_v \in \text{span}\{x_{\tilde{S}}(f)\}$, which means:

$$\text{span}\{x_u - x_v\}_{\{u,v\} \in \mathcal{S}} \subseteq \text{span}\{x_u - x_v\}_{u,v \in \tilde{S}} \subseteq \text{span}\{x_{\tilde{S}}(f)\} \implies \Pi_{\tilde{S}} \succeq P_S = P_S^2.$$

Consequently, $(x_u - x_v)^T \Pi_{\tilde{S}}(x_u - x_v) \geq (x_u - x_v)^T P_S^2(x_u - x_v) = \|P_S(x_u - x_v)\|^2$. \square

We wrap up this section with the following theorem.

Theorem 16. Given $x \in \text{Lasserre}_{r'}(V)$ and a set of seed edges $\mathcal{S} \subseteq \binom{V}{2}$ with projection matrices P_S, P_S^\perp as in Notation 14; let $T \subset V$ be the set returned by Algorithm 1 and y^S be the embedding as described in Definition 8. Then the following bounds hold:

$$\frac{\Phi_T}{\Phi^{\text{SDP}}} \leq \frac{1}{\Phi^{\text{SDP}}} \frac{\sum_{u < v} C_{u,v} \|y_u^S - y_v^S\|_1}{\sum_{u < v} D_{u,v} \|y_u^S - y_v^S\|_1} \leq \left(1 - \frac{\sum_{u < v} D_{u,v} \|P_S^\perp(x_u - x_v)\|^2}{\sum_{u < v} D_{u,v} \|x_u - x_v\|^2}\right)^{-1} \quad (7)$$

where $\Phi^{\text{SDP}} \stackrel{\text{def}}{=} \frac{\sum_{u < v} C_{u,v} \|x_u - x_v\|^2}{\sum_{u < v} D_{u,v} \|x_u - x_v\|^2}$.

Proof. $\Phi_T \leq \frac{\sum_{u < v} C_{u,v} \|y_u^S - y_v^S\|_1}{\sum_{u < v} D_{u,v} \|y_u^S - y_v^S\|_1}$ follows from Lemma 9. Claim 10 and Lemma 15 together imply

$$\begin{aligned} \frac{\sum_{u < v} C_{u,v} \|y_u^S - y_v^S\|_1}{\sum_{u < v} D_{u,v} \|y_u^S - y_v^S\|_1} &\leq \frac{\sum_{u < v} C_{u,v} \|x_u - x_v\|^2}{\sum_{u < v} D_{u,v} \|x_u - x_v\|^2 - \sum_{u < v} D_{u,v} \|P_S^\perp(x_u - x_v)\|^2} \\ &= \Phi^{\text{SDP}} \left(1 - \frac{\sum_{u < v} D_{u,v} \|P_S^\perp(x_u - x_v)\|^2}{\sum_{u < v} D_{u,v} \|x_u - x_v\|^2}\right)^{-1}. \end{aligned} \quad \square$$

3.3 Choosing Seed Edges

Notation 17. Given $x = [x_T \in \mathbb{R}^T]$ and $D : \binom{V}{2} \rightarrow \mathbb{R}_+$, let $\hat{X} \in \mathbb{R}^{T, \binom{V}{2}}$ be the following matrix whose columns are associated with vertex pairs: $\hat{X} \stackrel{\text{def}}{=} [\sqrt{D_{u,v}}(x_u - x_v)]_{\{u,v\} \in \binom{V}{2}}$.

Observe that $\|\hat{X}\|_F^2 = \sum_{u < v} D_{u,v} \|x_u - x_v\|^2$. Since $\mathcal{S} \subseteq \binom{V}{2}$, the matrix $\hat{X}_\mathcal{S}$ is well defined. Moreover there is a strong connection between $\hat{X}_\mathcal{S}^\Pi$ and P_S , which we formalize next:

Claim 18. $P_S \succeq (\hat{X}_\mathcal{S})^\Pi$. Furthermore if $\mathcal{S} \subseteq \text{support}(D)$ then $P_S = (\hat{X}_\mathcal{S})^\Pi$.

Proof. Recall that $\mathcal{S} \subseteq \binom{V}{2}$ and P_S represents $\text{span}\{x_u - x_v\}_{\{u,v\} \in \mathcal{S}}$, which contains every column of $\hat{X}_\mathcal{S} = [\sqrt{D_{u,v}}(x_u - x_v)]_{\{u,v\} \in \mathcal{S}}$. \square

After substituting the Notation 17, the upper bound in Theorem 16 becomes $\left(1 - \frac{\|(\hat{X}_\mathcal{S})^\perp \hat{X}\|_F^2}{\|\hat{X}\|_F^2}\right)^{-1}$..

One way to think about $\|(\hat{X}_\mathcal{S})^\perp \hat{X}\|_F^2$ is in terms of column based matrix reconstruction. If we were to express each column of \hat{X} as a linear combination of only r -columns of \hat{X} , what is the minimum reconstruction error (in terms of Frobenius norm) we can achieve? Without the restriction of choosing only columns, this question becomes easy to answer: Sum of all but largest r eigenvalues of Gram matrix, $\hat{X}^T \hat{X}$. We formalize this in Claim 20.

Notation 19. Let $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m \geq 0$ be the eigenvalues of $\hat{X}^T \hat{X}$ in descending order.

Claim 20. For any seed set $\mathcal{S} \subseteq \binom{V}{2}$ with $|\mathcal{S}| = r - 1$, $\|(\hat{X}_{\mathcal{S}})^\perp \hat{X}\|_F^2 \geq \sum_{j \geq r+1} \sigma_j$.

Proof. Follows from $\text{rank}(\hat{X}_{\mathcal{S}}^\Pi) \leq |\mathcal{S}| = r$ and Courant-Fischer Theorem. \square

In [GS12b], it was shown that choosing $\sim \frac{r}{\varepsilon}$ many columns suffice to decrease the error within a $(1 + \varepsilon)$ -factor of this lower bound and this is essentially the best possible up to low order terms.

Theorem 21 ([GS12b]). For any positive integer r and positive real ε , there exists $(\frac{r}{\varepsilon} + r - 1)$ columns of \hat{X} , \mathcal{S} , such that $\|(\hat{X}_{\mathcal{S}})^\perp \hat{X}\|_F^2 \leq (1 + \varepsilon) \sum_{j \geq r+1} \sigma_j$. Furthermore there exists an algorithm to find such \mathcal{S} in time $\text{poly}(n)$ (recall \hat{X} has $\binom{n}{2} = O(n^2)$ columns).

Our seed selection procedure is presented in Algorithm 2. We bound $\sum_{j \geq r+1} \sigma_j$ in Theorem 22, whose proof is given in Appendix A. Main approximation algorithm combining Algorithms 1 and 2 is presented in Algorithm 3 with its analysis in Corollary 23.

Theorem 22. Let $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$ be the generalized eigenvalues of Laplacian matrices for the cost and demand graphs. Then for \hat{X} being the matrix defined in Notation 17, the following bound holds:

$$\frac{\sum_{j \geq r+1} \sigma_j}{\|\hat{X}\|_F^2} \leq \frac{\Phi^{\text{SDP}}}{\lambda_{r+1}}.$$

We put everything together in the following corollary.

Corollary 23. Given $C, D : \binom{V}{2} \rightarrow \mathbb{R}_+$ representing cost and demand graphs, let $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$ be their generalized eigenvalues in ascending order. For any positive integer r and real $\varepsilon > 0$, on input C, D and $r' \stackrel{\text{def}}{=} \frac{r}{\varepsilon} + r - 1$, Algorithm 3 outputs a subset $T \subset V$ whose sparsity, Φ_T , is bounded by:

$$\Phi_T \leq \Phi^* \left(1 - (1 + \varepsilon) \frac{\Phi^*}{\lambda_{r+1}} \right)^{-1} \text{ provided that } (1 + \varepsilon) \frac{\Phi^*}{\lambda_{r+1}} < 1.$$

Furthermore using the SDP solver from [GS12a], the running time can be decreased to $2^{O(r')} \text{poly}(n)$.

Proof. Follows from Theorems 16, 21 and 22. \square

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Algorithm 1 $T = \text{ROUND}(C, D, x, \mathcal{S})$: Seed based rounding in time $2^{O(r')} \text{poly}(n)$. Sparsity of its output is bounded in Theorem 16.

Input: • $C, D : \binom{V}{2} \rightarrow \mathbb{R}_+$; $x \in \text{Lasserre}_{2r'+2}(V)$ and seed set $\mathcal{S} \subseteq \binom{V}{2}$ with $|\mathcal{S}| \leq r'$.

Output: • A set $T \subset V$ representing an approximation for SPARSEST CUT problem.

Procedure: 1. $\tilde{\mathcal{S}} \leftarrow \{u \in V \mid \text{exists } v \text{ such that } \{u, v\} \in \mathcal{S}\} \subseteq V$.

2. For each $f : \tilde{\mathcal{S}} \rightarrow \{0, 1\}$,

(a) Let $p^f : [n] \rightarrow V$ be an ordering of V so that $\langle x_{\tilde{\mathcal{S}}(f)}, x_{p^f(1)} \rangle \leq \dots \leq \langle x_{\tilde{\mathcal{S}}(f)}, x_{p^f(n)} \rangle$.

(b) For each $i \in [n]$, let $T(f, i) \leftarrow \{p^f(1), p^f(2), \dots, p^f(i)\}$.

3. $T \leftarrow \argmin_{f: \tilde{\mathcal{S}} \rightarrow \{0, 1\}, i \in [n]} \Phi_{T(f, i)}$.

Algorithm 2 $\mathcal{S} = \text{SELECT-SEEDS}(D, x)$: Seed selection in time $\text{poly}(n)$.

Input: • $x \in \text{Lasserre}_{2r'+2}(V)$ and $D : \binom{V}{2} \rightarrow \mathbb{R}_+$ as the demand graph.

Output: • $\mathcal{S} \subseteq \binom{V}{2}$ with $|\mathcal{S}| \leq r'$ as a set of seed edges.

Procedure: 1. Let $\hat{X} \leftarrow [\sqrt{D_{u,v}}(x_u - x_v)]_{\{u,v\} \in \binom{V}{2}}$.

2. Use the algorithm from [GS11] to choose r' -columns, $\mathcal{S} \subseteq \binom{V}{2}$, of matrix \hat{X} and return \mathcal{S} .

Algorithm 3 $T = \text{APPROXIMATE-SC}(C, D, r')$: Main algorithm for approximating SPARSEST CUT. Sparsity of the output is bounded in Corollary 23. A naïve implementation will run in time $n^{O(r')}$. However this algorithm exactly fits into the local rounding framework introduced in [GS12a], therefore we can use the faster solver from [GS12a] to decrease the running time to $2^{O(r')} \text{poly}(n)$.

Input: • $C, D : \binom{V}{2} \rightarrow \mathbb{R}_+$ as the cost and demand graphs, respectively.

Output: • A set $T \subset V$ representing an approximation for SPARSEST CUT problem.

Procedure: 1. Compute a (near-)optimal solution, x , to the following SDP:

$$\begin{aligned} \min \quad & \sum_{u < v} C_{u,v} \|x_u - x_v\|^2 \\ \text{st} \quad & \sum_{u < v} D_{u,v} \|x_u - x_v\|^2 = 1, \quad \|x_\emptyset\|^2 > 0, \quad x \in \text{Lasserre}_{2r'+2}(V). \end{aligned} \tag{8}$$

2. Let $\mathcal{S} \leftarrow \text{SELECT-SEEDS}(D, x)$ (Algorithm 2).

3. Let $T \leftarrow \text{ROUND}(C, D, x, \mathcal{S})$ (Algorithm 1). Return T .

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A Omitted Proofs from Section 3

Lemma 24 (Restatement of Lemma 9). *Given a set of vertices V , a collection of vectors $[y_u \in \mathbb{R}^\Upsilon]_{u \in V}$ representing an embedding of V , the following holds. For any $C, D : \binom{V}{2} \rightarrow \mathbb{R}_+$ being the edge weights of graphs G and H , respectively:*

$$\min_{\substack{f \in \Upsilon, \\ \tau \in \mathbb{R}}} \Phi_{T(f, \tau)} \leq \frac{\sum_{u < v} C_{u,v} \|y_u - y_v\|_1}{\sum_{u < v} D_{u,v} \|y_u - y_v\|_1}.$$

Here $T(f, \tau) \stackrel{\text{def}}{=} \{u \in V \mid y_u(i) \geq \tau\}$ represents the threshold cut along coordinate $f \in \Upsilon$.

Proof of Lemma 9. Let $\phi \stackrel{\text{def}}{=} \min_{\substack{f \in \Upsilon, \\ \tau \in \mathbb{R}}} \Phi_{T(f, \tau)}$. For any $f \in \Upsilon$, let $\delta_f \stackrel{\text{def}}{=} \max_{a,b} |y_a(f) - y_b(f)| = \max_b y_b(f) - \min_a y_a(f)$ and $\Delta \stackrel{\text{def}}{=} \sum_f \delta_f$.

Consider the following randomized process. Choose $f \in \Upsilon$ with probability proportional to δ_f and then sample a threshold $\tau \in [\min_a y_a(f), \max_b y_b(f)]$. Then:

$$\mathbb{E}_{f, \tau} \left[|\mathbb{1}_{T(f, \tau)}(u) - \mathbb{1}_{T(f, \tau)}(v)| \right] = \sum_{f \in \Upsilon} \frac{\delta_f}{\Delta} \frac{|y_u(f) - y_v(f)|}{\delta_f} = \frac{1}{\Delta} \|y_u - y_v\|_1.$$

Moreover, for any f and τ , by definition of ϕ :

$$\sum_{u < v} C_{u,v} |\mathbb{1}_{T(f, \tau)}(u) - \mathbb{1}_{T(f, \tau)}(v)| \geq \phi \sum_{u < v} D_{u,v} |\mathbb{1}_{T(f, \tau)}(u) - \mathbb{1}_{T(f, \tau)}(v)|.$$

Putting it all together:

$$\frac{\sum_{u < v} C_{u,v} \|y_u - y_v\|_1}{\sum_{u < v} D_{u,v} \|y_u - y_v\|_1} = \frac{\mathbb{E}_{f, \tau} \left[\sum_{u < v} C_{u,v} |\mathbb{1}_{T(f, \tau)}(u) - \mathbb{1}_{T(f, \tau)}(v)| \right]}{\mathbb{E}_{f, \tau} \left[\sum_{u < v} D_{u,v} |\mathbb{1}_{T(f, \tau)}(u) - \mathbb{1}_{T(f, \tau)}(v)| \right]} \geq \phi. \quad \square$$

Theorem 25 (Restatement of Theorem 22). *Let $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$ be the generalized eigenvalues of Laplacian matrices for the cost and demand graphs. Then the following bound holds:*

$$\frac{\sum_{j \geq r+1} \sigma_j}{\|\hat{X}\|_F^2} \leq \frac{\Phi^{\text{SDP}}}{\lambda_{r+1}},$$

where \hat{X} is the matrix given in Notation 17.

Proof of Theorem 22. Since the claimed bound is scale independent, we may assume $\|\hat{X}\|_F^2 = 1$ without loss of generality.

Throughout the proof, we will use the following matrices:

- $X \stackrel{\text{def}}{=} [x_u]_{u \in V} \in \mathbb{R}^{\Upsilon, V}$,
- $B_C \in \mathbb{R}^{\binom{V}{2}, V}$ is the following edge-node incidence matrix of the cost graph whose columns and rows are associated with vertices and edges, respectively. Its entry at column $c \in V$ and row $\{a, b\} \in \binom{V}{2}$ with $a < b$ (assuming some consistent ordering of V) is given by:

$$(B_C)_{\{a, b\}, c} \stackrel{\text{def}}{=} \sqrt{C_{u,v}} \begin{cases} 1 & \text{if } c = a, \\ -1 & \text{if } c = b, \\ 0 & \text{else.} \end{cases}$$

- $B_D \in \mathbb{R}^{\binom{V}{2}, V}$ is defined similarly for the demand graph.
- L_C, L_D are the Laplacian matrices for cost and demand graphs, respectively.
- $(L_D)^\dagger$ is the pseudo-inverse of L_D .

The following identities are trivial:

$$\hat{X} = X(B_D)^T; \quad L_C = B_C^T B_C; \quad L_D = B_D^T B_D.$$

Moreover

$$\sum_{u < v} C_{u,v} \|x_u - x_v\|^2 = \|XB_C^T\|_F^2 = \text{Tr}(XL_C X^T) = \Phi^{\text{SDP}} \text{Tr}(XL_D X^T) = \Phi^{\text{SDP}}$$

by our assumption that $\|\hat{X}\|_F^2 = \text{Tr}(XL_D X^T) = 1$.

Since $(L_D)^\Pi$ is a projection matrix and $L_C \succeq 0$, we have $L_C \succeq (L_D)^\Pi L_C (L_D)^\Pi$. Substituting the identity $(L_D)^\Pi = L_D(L_D)^\dagger = (L_D)^\dagger L_D$ into this lower bound, we have:

$$\begin{aligned} L_C &\succeq L_D L_D^\dagger L_C L_D^\dagger L_D = (B_D)^T \left[B_D \underbrace{L_D^\dagger L_C L_D^\dagger}_{\stackrel{\text{def}}{=} Z} (B_D)^T \right] B_D, \\ \implies \Phi^{\text{SDP}} = \text{Tr}(XL_C X^T) &\geq \text{Tr} \left\{ X(B_D)^T [B_D Z (B_D)^T] B_D X^T \right\} \\ &= \text{Tr} \left\{ \hat{X} [B_D Z (B_D)^T] \hat{X}^T \right\} \end{aligned}$$

The null space of $\hat{X} = X(B_D)^T$ contains the null space of $(B_D)^T$. In particular, non-zero eigenvectors of $\hat{X}^T \hat{X}$ are contained in the span of $B_D(B_D)^T$. Using von Neumann-Birkhoff Theorem, we obtain:

$$\geq \sum_j \sigma_j \lambda_j \geq \lambda_{r+1} \sum_{j \geq r+1} \sigma_j. \quad \square$$

B Using Subspace Enumeration for UNIFORM SPARSEST CUT

Throughout this section, we will assume that the cost graph with weights $C : \binom{V}{2} \rightarrow \mathbb{R}_+$ is 1-regular. Since G is regular, definitions of uniform sparsest cut / normalized cut and edge expansion / conductance coincide. Thus we will focus only on UNIFORM SPARSEST CUT which we denote by ϕ^* .

The following theorem is adapted from [AL08] for our setting:

Theorem 26 (Cut Improvement, see [AL08]). *For any $x^* \in \{0, 1\}^V$, given $x \in \{0, 1\}^V$ satisfying*

$$0 < \|x\|_1 \leq \frac{n}{2} \text{ and } \frac{\langle x, x^* \rangle}{\|x^*\|_1} > \frac{\|x\|_1}{n}$$

in polynomial time one can find $y \in \{0, 1\}^V$ whose edge expansion is within a factor

$$\leq \frac{1 - \|x\|_1/n}{\langle x, x^* \rangle / \|x^*\|_1 - \|x\|_1/n}$$

of x^ 's edge expansion.*

The following lemma is adapted from [ABS10]:

Theorem 27 (Eigenspace Enumeration, see [ABS10]). *In time $2^{O(r)}n^{O(1)}$, there exists an algorithm which outputs a set $X \subseteq \{0,1\}^V$ that contains some $x \in X$ with following property: There exists $x^* \in \{0,1\}^V$ with:*

$$\frac{\|x - x^*\|_1}{\|x^*\|_1} \leq \frac{8}{\lambda_r} \phi^*.$$

Combining these two, we obtain the following:

Corollary 28. *For any positive integer r , if r^{th} smallest eigenvalue of Laplacian matrix for cost graph satisfies $\lambda_r > 8\phi^*$ where ϕ^* is the UNIFORM SPARSEST CUT value, then in time $n^{O(1)}2^{O(r)}$ one can find $y \in \{0,1\}^V$ whose uniform sparsity is bounded by:*

$$\frac{2\phi^*}{1 - 8\frac{\phi^*}{\lambda_r}}.$$